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Fixed Point Theorems of Compatible maps on Complete G-Metric space

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Abstract: In this paper common fixed point theorems of two compatible self-mappings on a complete G-Metric space is proved using rectangular inequality of G-Metric space.

Keywords: Compatible maps, Rectangular inequality, G-Metric space.

1. Introduction and preliminaries

Banach [3] proved a fixed-point theorem "Let (X, d) be a complete metric space. If T satisfies d(Tx, Ty) $\leq$ kd(x,y) for each x, y in X where 0 <k < 1, then T has a unique fixed point in X ." Dhage [4] introduced D-metric spaces. Geometrically, a D-metric D(x, y, z) represents the perimeter of the triangle with vertices x, y and z in R× R . Mustafa and Sims [6] proved that most of the results of Dhage's D-metric spaces were not true. So, they introduced a new generalized metric space and called it as G-metric spaces. Further G.Jungck defined compatibility of pair of self mappings of a metric space.Some basic definitions and results in G-metric spaces which are useful for proving the main result are as follows.

Mustafa and Sims defined G-metric spaces as a generalization of metric space.

Definition 1.1 [7] "Let G: X × X × X $\to$ R^+ be a function on a non-empty X satisfying

(G-1) G(x, y, z) = 0 if x = y = z,

(G-2) 0 < G(x, x, y) for all x, y ∈ X with x $\neq$ y,

(G-3) G (x, x, y) $\leq$ G (x, y, z) for all x, y, z ∈ X with z $\neq$ y,

(G-4) G(x, y, z) = G(x, z, y) = G(y, z, x) = … (symmetry in all three variables),

(G-5) G(x, y, z) $\leq$ G(x, a, a) + G(a, y, z) for all x, y, z, a ∈ X, (rectangle inequality).

The function G is called a generalized metric or more specifically, a G-metric on Xand the pair (X, G) is called a G-metric space."

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Definition 1.2 [7] "A sequence $\{x_n\}$ of points in G-metric space X is said to be G-convergent to x if $\lim_{m,n\to\infty} G(x, x_n, x_m) = 0$; i.e. for each $\in > 0$ there exists a positive integer N_1 such that $G(x, x_n, x_m) < \in$ for all $m, n \geq N_1$. We say x is the limit of the sequence and write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$."

Theorem 1.3 [7] "The following are equivalent in a G-metric space:

(i) $\{x_n\}$ is G-convergent to x;

(ii) $G(x_n, x_n, x) \to 0$ as $n \to \infty$;

(iii) $G(x_n, x, x) \to 0$ as $n \to \infty$;

(iv) $G(x_m, x_n, x) \to 0$ as $m, n \to \infty$."

Definition 1.4 [7]"Let (X, G) be a G-metric space. A sequence $\{x_n\}$ is called G-Cauchy if, for each $\in > 0$ there exists a positive integer N_1 such that

$$G(x_n, x_m, x_l) < \in \text{ for all } n, m, l \geq N_1."$$

Theorem 1.5 [7]" The following are equivalent in a G-metric space :

(i) the sequence $\{x_n\}$ is G-Cauchy,

(ii) for each $\in > 0$ there exists an N such that $G(x_n, x_m, x_l) < \in$ for all $n, m, l \geq N_1$."

Theorem 1.6 [7]"The function G(x, y, z) is jointly continuous in all three of its variables in a G-metric space."

Definition 1.7 [7]"A G-metric space (X, G) is called a symmetric G-metric space if $G(x, y, y) = G(y, x, x)$ for all x, y in X."

Theorem 1.8 [7]"Every G-metric space (X, G) defines a metric space (X, d_G) by

$d_G(x, y) = G(x, y, y) + G(y, x, x)$ for all x, y in X.

If (X, G) is a symmetric G-metric space, then

$$d_G(x, y) = 2G(x, y, y) \text{ for all } x, y \text{ in } X.$$

However, if (X, G) is not symmetric, then it follows from the G-metric properties that

$\frac{3}{2} G(x, y, y) \leq d_G(x, y) \leq 3G(x, y, y)$ for all x, y in X."

Theorem 1.9 [7]"A G-metric space (X, G) is G-complete if and only if (X, d_G) is a complete metric space."

Theorem 1.10 [7]"Let (X, G) be a G-metric space. Then, for any x, y, z, a in X, it follows that:

(i) if $G(x, y, z) = 0$, then $x = y = z$;

(ii) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$;

(iii) $G(x, y, y) \leq 2G(y, x, x)$;

(iv) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$;

(v) $G(x, y, z) \leq {}^{2}/_{3}\,(G(x, a, a) + G(y, a, a) + G(z, a, a))$."

Definition 1.11 [7]"Let f and g be single-valued self-mappings on a set X. If $w = fx = gx$ for some $x \in X$, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g."

Definition 1.12 [5] "A pair (f, g) of self-mappings of a metric space (X, d) is said to be compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X$."

2. Main Results

Theorem 2.1 Let f and g be compatible self-maps of a complete G-metric space (X, G) satisfying

(i) $f(X) \subseteq g(X)$;

(ii) $$G(fp, fq, fr) \leq \alpha\, G(fp, gq, gr) + \beta\, G(gp, fq, gr) + \gamma\, G(gp, gq, fr) + \delta\, G(gq, gq, fq)$$

for all $p, q, r \in X$ and $\alpha, \beta, \gamma, \delta \geq 0$ with $0 \leq \alpha + 3\beta + 3\gamma + 3\delta < 1$;

(iii) one of f or g is continuous.

Then f and g have a unique common fixed point in X.

Proof. Let α_0 be an arbitrary point in X. Since $f(X) \subseteq g(X)$, one can choose a point α_1 in X such that $f\alpha_0 = g\alpha_1$, In general one can choose α_{n+1} such that

$t_n = f\alpha_n = g\alpha_{n+1},\ n = 0,1,2,\ldots.$

From (ii), we have

$$G(f\alpha_n, f\alpha_{n+1}, f\alpha_{n+1}) \leq \alpha\, G(f\alpha_n, g\alpha_{n+1}, g\alpha_{n+1}) + \beta\, G(g\alpha_n, f\alpha_{n+1}, g\alpha_{n+1})$$
$$+ \gamma\, G(g\alpha_n, g\alpha_{n+1}, f\alpha_{n+1}) + \delta\, G(g\alpha_{n+1}, g\alpha_{n+1}, f\alpha_{n+1})$$
$$= \alpha\, G(f\alpha_n, f\alpha_n, f\alpha_n) + \beta\, G(f\alpha_{n-1}, f\alpha_{n+1}, f\alpha_n)$$

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$$+ \gamma\, G(f\alpha_{n-1}, f\alpha_n, f\alpha_{n+1}) + \delta\, G(f\alpha_n, f\alpha_n, f\alpha_{n+1})$$

$$\leq \alpha\, G(f\alpha_n, f\alpha_n, f\alpha_n) + \beta\, G(f\alpha_{n-1}, f\alpha_{n+1}, f\alpha_n)$$

$$+ \gamma\, G(f\alpha_{n-1}, f\alpha_n, f\alpha_{n+1}) + \delta\, G(f\alpha_{n-1}, f\alpha_n, f\alpha_{n+1})$$

$$= (\beta + \gamma + \delta)\, G(f\alpha_{n-1}, f\alpha_n, f\alpha_{n+1}). \qquad (2.1)$$

Using rectangular inequality of G-metric space, we have

$$G(f\alpha_{n-1}, f\alpha_n, f\alpha_{n+1}) \leq G(f\alpha_{n-1}, f\alpha_n, f\alpha_n) + G(f\alpha_n, f\alpha_n, f\alpha_{n+1})$$

$$\leq G(f\alpha_{n-1}, f\alpha_n, f\alpha_n) + 2G(f\alpha_n, f\alpha_{n+1}, f\alpha_{n+1}). \qquad (2.2)$$

Using (2.1) in (2.2) we have

$$(1-2\beta - 2\gamma - 2\delta)\, G(f\alpha_n, f\alpha_{n+1}, f\alpha_{n+1}) \leq (\beta + \gamma + \delta)\, G(f\alpha_{n-1}, f\alpha_n, f\alpha_n)$$

i.e. $G(f\alpha_n, f\alpha_{n+1}, f\alpha_{n+1}) \leq \frac{(\beta+\gamma+\delta)}{1-2(\beta+\gamma+\delta)}\, G(f\alpha_{n-1}, f\alpha_n, f\alpha_n)$

$$= q\, G(f\alpha_{n-1}, f\alpha_n, f\alpha_n), \text{where } q = \frac{(\beta+\gamma+\delta)}{1-2(\beta+\gamma+\delta)} < 1.$$

Repeatedly, we have

$$G(f\alpha_n, f\alpha_{n+1}, f\alpha_{n+1}) \leq q^n G(f\alpha_0, f\alpha_1, f\alpha_1).$$

Therefore, for all n, m $\in$ N, n < m, we have by rectangular inequality,

$$G(t_n, t_m, t_m) \leq G(t_n, t_{n+1}, t_{n+1}) + G(t_{n+1}, t_{n+2}, t_{n+2}) + \ldots + G(t_{m-1}, t_m, t_m)$$

$$\leq (q^n + q^{n+1} + \ldots + q^{m-1})\, G(t_0, t_1, t_1).$$

$$\leq \frac{q^n}{(1-q)} G(t_0, t_1, t_1).$$

As n, m $\to\infty$, we have $\lim_{m,n\to\infty} G(t_n, t_m, t_m) = 0$.

Thus $\{t_n\}$ is a G-Cauchy sequence in complete G-metric space X, therefore there exists a point s $\in$ X such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} f\alpha_n = \lim_{n\to\infty} g\alpha_{n+1} = s$.

Since the mapping f or g is continuous and let it be g.

Therefore $\lim_{n\to\infty} gf\alpha_n = \lim_{n\to\infty} gg\alpha_{n+1} = gs$.

Further, f and g are compatible, therefore

$$\lim_{n\to\infty} G(fg\alpha_n, gf\alpha_n, gf\alpha_n) = 0 \text{ implies that} \lim_{n\to\infty} fg\alpha_n = gs.$$

Setting p = $g\alpha_n$, q = α_n and r = α_n, in (ii), we have

$$G(fg\alpha_n, f\alpha_n, f\alpha_n) \le \alpha\, G(fg\alpha_n, g\alpha_n, g\alpha_n) + \beta\, G(gg\alpha_n, f\alpha_n, g\alpha_n) + \gamma\, G(gg\alpha_n, g\alpha_n, f\alpha_n) + \delta\, G(g\alpha_n, g\alpha_n, f\alpha_n).$$

Letting $n \to \infty$, we have gs = s.

Again from (ii), we have

$G(f\alpha_n, fs, fs) \le \alpha\, G(f\alpha_n, gs, gs) + \beta\, G(g\alpha_n, fs, gs)$

$+ \gamma\, G(g\alpha_n, gs, fs) + \delta\, G(gs, gs, fs)$.

Letting $n \to \infty$, we have fs = s.

Therefore, fs = gs = s i.e. s is a common fixed point of f and g.

Uniqueness: We assume that $s_1 (\neq s)$ be another common fixed point of f and g .

Then $G(s, s_1, s_1) > 0$ and

$G(s, s_1, s_1) = G(fs, fs_1, fs_1)$

$\le \alpha\, G(fs, gs_1, gs_1) + \beta\, G(gs, fs_1, gs_1) + \gamma\, G(gs, gs_1, fs_1)$

$+ \delta\, G(gs_1, gs_1, fs_1)$

$= (\alpha + \beta + \gamma)\, G(s, s_1, s_1) < G(s, s_1, s_1)$, a contradiction,

which proves that $s = s_1$ and hence uniqueness.

Theorem 2.2 Let f and g be compatible self-maps of a complete G-metric space (X, G) satisfying

(i) $f(X) \subseteq g(X)$;

(ii) $G(fp, fq, fr) \le k \max\left\{\begin{matrix} G(fp, gq, gr), G(gp, fq, gr), \\ G(gp, gq, fr), G(gq, gq, fq) \end{matrix}\right\}$,

for all p, q, r $\in$ X and $k \in [0, \frac{1}{3})$;

(iii) one of f or g is continuous.

Then f and g have a unique common fixed point in X.

Proof. Let α_0 be an arbitrary point in X. Since $f(X) \subseteq g(X)$, one can choose a point α_1 in X such that $f\alpha_0 = g\alpha_1$, In general one can choose α_{n+1} such that

$t_n = f\alpha_n = g\alpha_{n+1}$, n = 0,1,2,….

From (ii), we have

$$G(f\alpha_n, f\alpha_{n+1}, f\alpha_{n+1}) \leq k \max\begin{Bmatrix} G(f\alpha_n, g\alpha_{n+1}, g\alpha_{n+1}), G(g\alpha_n, f\alpha_{n+1}, g\alpha_{n+1}), \\ G(g\alpha_n, g\alpha_{n+1}, f\alpha_{n+1}), G(g\alpha_{n+1}, g\alpha_{n+1}, f\alpha_{n+1}) \end{Bmatrix}$$

$$= k \max\begin{Bmatrix} G(f\alpha_n, f\alpha_n, f\alpha_n), G(f\alpha_{n-1}, f\alpha_{n+1}, f\alpha_n), \\ G(f\alpha_{n-1}, f\alpha_n, f\alpha_{n+1}), G(f\alpha_n, f\alpha_n, f\alpha_{n+1}) \end{Bmatrix}$$

$$= k \max\{0, G(f\alpha_{n-1}, f\alpha_{n+1}, f\alpha_n), G(f\alpha_n, f\alpha_n, f\alpha_{n+1})\}$$

$$= k \max\{G(f\alpha_{n-1}, f\alpha_n, f\alpha_{n+1}), G(f\alpha_n, f\alpha_n, f\alpha_{n+1})\}$$

$$= k\, G(f\alpha_{n-1}, f\alpha_n, f\alpha_{n+1}).$$

By rectangular inequality of G-metric space, we have

$$G(f\alpha_{n-1}, f\alpha_n, f\alpha_{n+1}) \leq G(f\alpha_{n-1}, f\alpha_n, f\alpha_n) + G(f\alpha_n, f\alpha_n, f\alpha_{n+1})$$

$$\leq G(f\alpha_{n-1}, f\alpha_n, f\alpha_n) + 2\, G(f\alpha_n, f\alpha_{n+1}, f\alpha_{n+1}).$$

Therefore from above inequality, we have

$$G(f\alpha_n, f\alpha_{n+1}, f\alpha_{n+1}) \leq \frac{k}{(1-2k)} G(f\alpha_{n-1}, f\alpha_n, f\alpha_n),$$

$$= qG(f\alpha_{n-1}, f\alpha_n, f\alpha_n), \text{ where } q = \frac{k}{(1-2k)} < 1.$$

Continuingly, we have

$$G(f\alpha_n, f\alpha_{n+1}, f\alpha_{n+1}) \leq q^n G(f\alpha_0, f\alpha_1, f\alpha_1).$$

Therefore, for all n, m $\in$ N, n < m, we have by rectangular inequality

$$G(t_n, t_m, t_m) \leq G(t_n, t_{n+1}, t_{n+1}) + G(t_{n+1}, t_{n+2}, t_{n+2}) + G(t_{n+2}, t_{n+3}, t_{n+3}) + \ldots + G(t_{m-1}, t_m, t_m)$$

$$\leq (q^n + q^{n+1} + \ldots + q^{m-1})\, G(t_0, t_1, t_1).$$

$$\leq \frac{q^n}{(1-q)} G(t_0, t_1, t_1).$$

As n, m $\to\infty$, we have $\lim_{m,n\to\infty} G(t_n, t_m, t_m) = 0$.

Thus $\{t_n\}$ is a G-Cauchy sequence in complete G-metric space X, therefore, there exists a point s $\in$ X such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} f\alpha_n = \lim_{n\to\infty} g\alpha_{n+1} = s$.

Since the mapping f or g is continuous and let it be g , therefore

$$\lim_{n\to\infty} gf\alpha_n = \lim_{n\to\infty} gg\alpha_{n+1} = gs.$$

Further, f and g are compatible,therefore,

$\lim_{n\to\infty} G(fg\alpha_n, gf\alpha_n, gf\alpha_n) = 0$ implies that $\lim_{n\to\infty} fg\alpha_n = gs$.

Setting $p = g\alpha_n$, $q = \alpha_n$ and $r = \alpha_n$ in (ii), we have

$$G(fg\alpha_n, f\alpha_n, f\alpha_n) \le k \max \begin{Bmatrix} G(fg\alpha_n, g\alpha_n, g\alpha_n), G(gg\alpha_n, f\alpha_n, g\alpha_n), \\ G(gg\alpha_n, g\alpha_n, f\alpha_n), G(g\alpha_n, g\alpha_n, f\alpha_n) \end{Bmatrix}.$$

Letting as $n \to \infty$, we have

$$G(gs, s, s) \le k \max \{G(gs, s, s), G(gs, s, s), G(gs, s, s)\} \text{implies, } gs = s.$$

Again from (ii), we have

$$G(f\alpha_n, fs, fs) \le k \max \begin{Bmatrix} G(f\alpha_n, gs, gs), G(g\alpha_n, fs, gs), \\ G(g\alpha_n, gs, fs), G(gs, gs, fs) \end{Bmatrix}.$$

Letting as $n \to \infty$, we have

$$G(s, fs, fs) \le k \max \begin{Bmatrix} G(s, s, s), G(s, fs, s), \\ G(s, s, fs), G(s, s, fs) \end{Bmatrix}$$

$= k\, G(s, s, fs) \le 2k\, G(s, fs, fs)$ which is not possible.

And so, $G(s, fs, fs) = 0$, that is $fs = s$.

Therefore, $fs = gs = s$ i.e. s is a common fixed point of f and g.

Uniqueness: We assume that $s_1 (\neq s)$ be another common fixed point of f and g .

Then $G(s, s_1, s_1) > 0$ and

$$G(s, s_1, s_1) = G(fs, fs_1, fs_1)$$

$$\le k \max \begin{Bmatrix} G(fs, gs_1, gs_1), G(gs, fs_1, gs_1), \\ G(gs, gs_1, fs_1), G(gs_1, gs_1, fs_1) \end{Bmatrix}$$

$= k\, G(s, s_1, s_1) < G(s, s_1, s_1)$, a contradiction,

which proves that $s = s_1$ and hence uniqueness.

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